UNORIENTED HQFTS AND ITS UNDERLYING ALGEBRAS

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ABSTRACT. Turaev and Turner introduced a bijection between unoriented topological quantum field theories and extended Frobenius algebras. In this paper, we will show that there exists a bijective correspondence between unoriented $(1 + 1)$-dimensional homotopy quantum field theories and extended crossed group algebras.

1. INTRODUCTION

In [2], Atiyah introduced a mathematical definition of topological quantum field theories (TQFTs). A $(d + 1)$-TQFT assigns a module to each d-dimensional manifold and assigns a homomorphism of modules to each $(d + 1)$-dimensional cobordism. Abrams [1] showed that there is a bijective correspondence between oriented (1+1)-TQFTs and Frobenius algebras. Turaev [6] defined a concept of homotopy quantum field theories (HQFTs) with target X, where X is a connected topological space. An HQFT assigns a module and a homomorphism of modules to each "X-manifold" and "X-cobordism" respectively. For any group π, he constructed a bijective correspondence between oriented $(1 + 1)$-dimensional HQFTs with target X for $X = K(\pi, 1)$ and crossed π algebras in [6], where a crossed π algebra V is a Frobenius π-algebra endowed with a group homomorphism $\varphi \colon \pi \to \mathrm{Aut}(V)$. In [5] Staic and Turaev discussed $(1+1)$-dimensional HQFTs more generally. Turaev and Turner [8] showed that there exists a bijective correspondence between unoriented (1+1)-TQFTs and extended Frobenius algebras. An extended Frobenius algebra K is a Frobenius algebra endowed with an element $\theta \in K$ and a homomorphism $\Phi \colon K \to K$.

In this paper, we consider a group π such that $\alpha^2 = 1$ for any $\alpha \in \pi$, $X = K(\pi, 1)$ and unoriented $(1+1)$-dimensional HQFTs with target X. Moreover we introduce "extended crossed π algebra" L which consists of a Frobenius π algebra, a group homomorphism $\varphi \colon \pi \to \mathrm{Aut}(L)$, elements $\{\theta_\alpha \in L\}_{\alpha \in \pi}$ and a homomorphism $\Phi \colon L \to L$. We will show that there is a bijective correspondence between unoriented $(1+1)$-dimensional HQFTs with target X and extended crossed π algebras.

In Section 2, we recall definitions of HQFTs and some algebras introduced in [6] and will define unoriented HQFTs and extended crossed group algebras. At the end of this section, we introduce our main theorem (Theorem 2.12). In Section 3, we construct an extended crossed group algebra from an HQFT (A, τ). We call it underlying extended crossed group algebra of (A, τ). In Sections 4 and 5, we prove the main theorem. In Section 6, we give some examples.

Throughout this paper, the symbol R denotes a commutative ring with unit and the symbol π denotes a group.

2. Unoriented HQFTs and extended crossed group algebras

Here we will explain terminology used in this paper.

2.1. **Unoriented HQFTs.** In this subsection, we recall the definition of unoriented homotopy quantum field theories. An oriented homotopy quantum field theory is introduced by Turaev [6].

Definition 2.1. Let X be a CW-complex. We call X the *Eilenberg-Mac Lane space of type $K(\pi, 1)$* corresponding to a group π ($K(\pi, 1)$ space for short) if its homotopy group $\pi_n(X) = \pi$ with $n = 1$ and $\pi_n(X) = 0$ with $n \neq 1$. It is well known that such CW-complex is unique up to homotopy equivalence.

Definition 2.2 ([6]). A topological space is *pointed* if each of its connected components has a base point. A map between pointed spaces is a continuous map preserving their base points. Homotopies of such maps are always supposed to be constant on the base points.

Definition 2.3 ([6]). Let X be a $K(\pi, 1)$ space with a base point $x_0 \in X$. A pair (M, g_M) is called an *unoriented X-manifold* if M is a pointed closed unoriented manifold and g_M is a map from M to X. We call the map g_M the *characteristic map*. Since the spaces M and X are pointed, the map g_M sends the base points of all components of M to x_0. A disjoint union of unoriented X-manifolds and the empty set are also unoriented X-manifolds. An *unoriented X-homeomorphism* of unoriented X-manifolds $f \colon (M, g_M) \to (M', g_{M'})$ is a homeomorphism from M to M' sending the base points of M to those of M' such that $g_M = g_{M'} \circ f$.

Definition 2.4 ([6]). Let X be a $K(\pi, 1)$ space with a base point $x_0 \in X$. A triple (W, M_0, M_1) is called an *unoriented cobordism* when W is a compact manifold whose boundary is the disjoint union of pointed closed manifolds M_0 and M_1. An *unoriented X-cobordism* is a tuple (W, M_0, M_1, g) such that the triple (W, M_0, M_1) is an unoriented cobordism and that $g \colon W \to X$ is a map which sends the base points of M_0 and M_1 to $x_0 \in X$. We call the boundary M_0 the bottom base, M_1 the top base and the map g the characteristic map. An *unoriented X-homeomorphism* of X-cobordisms $f \colon (W, M_0, M_1, g) \to (W', M_0', M_1', g')$ is a homeomorphism from W to W' inducing unoriented X-homeomorphisms $M_0 \to M_0'$ and $M_1 \to M_1'$ such that $g = g' \circ f$.

Definition 2.5 ([6]). Fix an integer $d \geq 0$ and a path connected topological space X with base point $x \in X$. An *unoriented $(d+1)$-dimensional homotopy quantum field theory* (HQFT for short) (A, τ) over R with target X assigns

- a finitely generated projective R-module $A(M, g)$ ($A(M)$ for short) to any unoriented d-dimensional X-manifold (M, g),
- an R-isomorphism $f_\sharp \colon A(M, g) \to A(M', g')$ to any unoriented X-homeomorphism of d-dimensional X-manifolds $f \colon (M, g) \to (M', g')$,
- an R-homomorphism $\tau(W, g) \colon A(M_0, g|_{M_0}) \to A(M_1, g|_{M_1})$ to any $(d+1)$-dimensional X-cobordism (W, M_0, M_1, g).

Moreover these modules and homomorphisms should satisfy the following axioms:
    (1) for unoriented X-homeomorphisms of unoriented X-manifolds $f \colon M \to M'$ and $f' \colon M' \to M''$, we have $(f' \circ f)_\sharp = f'_\sharp \circ f_\sharp$,

(2) for unoriented d-dimensional X-manifolds M and N, there is a natural isomorphism $A(M \sqcup N) = A(M) \otimes A(N)$, where $M \sqcup N$ is the disjoint union of M and N,

(3) $A(\emptyset) = R$,

(4) for any unoriented X-cobordism W, the homomorphism $\tau(W)$ is natural with respect to unoriented X-homeomorphisms,

(5) if an unoriented $(d+1)$-dimensional X-cobordism (W, M_0, M_1, g) is the disjoint union of two unoriented $(d+1)$-dimensional X-cobordisms W_0 and W_1, then $\tau(W) = \tau(W_1) \otimes \tau(W_0)$,

(6) if an oriented $(d+1)$-dimensional X-cobordism (W, M_0, M_1, g) is obtained from two $(d+1)$-dimensional X-cobordism (W_0, M_0, N) and (W_1, N', M_1) by gluing along $f\colon N \to N'$, then $\tau(W) = \tau(W_1) \circ f_\sharp \circ \tau(W_0)$,

(7) for any unoriented d-dimensional X-manifold (M, g) and any continuous map $F\colon M \times [0,1] \to X$ such that $F|_{M \times 0} = F|_{M \times 1} = g$ and that $F(m \times [0,1]) = \{x\}$ for any base point m of M, we have $\tau(M \times [0,1], M \times 0, M \times 1, F) = \mathrm{id}_{A(M)}\colon A(M) \to A(M)$,

(8) for any unoriented $(d+1)$-dimensional X-cobordism (W, g), $\tau(W)$ is preserved under any homotopy of g relative to ∂W.

If two maps f and $f'\colon M \to X$ are homotopic, there is a natural isomorphism $A(M, f) \cong A(M, f')$. Hence we can suppose that $A(M, f)$ is preserved under any homotopy of f. Similary $\tau(W, g)$ is preserved under any homotopy of g (maybe not relative to ∂W). See [6].

2.2. **Extended crossed group algebras.** In this subsection, we recall some algebras which are introduced in [6] and define extended crossed group algebras.

Definition 2.6. An R-*algebra* L is a π-*algebra* over the ring R if L is an associative algebra over R endowed with a splitting $L = \bigoplus_{\alpha \in \pi} L_\alpha$ such that each L_α is a finitely generated projective R-module, that $L_\alpha L_\beta \subset L_{\alpha\beta}$ for any $\alpha, \beta \in \pi$, and that L has the unit element $1_L \in L_1$.

Let V and W be R-modules and $\eta\colon V \otimes W \to R$ be a bilinear form. The map η is non-degenerate if the two maps $d\colon V \to \mathrm{Hom}_R(W, R)$ defined by $d(v)(w) = \eta(v, w)$ and $s\colon W \to \mathrm{Hom}_R(V, R)$ defined by $s(w)(v) = \eta(v, w)$ are isomorphisms, where $v \in V$ and $w \in W$.

Definition 2.7 ([6]). A pair (L, η) is a *Frobenius* π-*algebra* over R if L is a π-algebra over R and $\eta\colon L_\alpha \otimes L_\beta \to R$ is an R-bilinear form such that

(1) $\eta(L_\alpha \otimes L_\beta) = 0$ if $\alpha\beta \neq 1$ and the restriction of η to $L_\alpha \otimes L_{\alpha^{-1}}$ is non-degenerate for any $\alpha \in \pi$,

(2) $\eta(ab, c) = \eta(a, bc)$ for any $a, b, c \in L$.

A Frobenius π-algebra with π a trivial group is called a *Frobenius algebra*. See [1].

For any Frobenius π-algebra (L, η), $\mathrm{Aut}(L)$ is a group which consists of algebra automorphisms preserving η.

Definition 2.8 ([6]). A triple (L, η, φ) is a *crossed* π-*algebra* over R if the pair (L, η) is a Frobenius π-algebra over R and $\varphi\colon \pi \to \mathrm{Aut}(L)$ is a group homomorphism satisfying the following axioms:

(1) for any $\beta \in \pi$, $\varphi_\beta := \varphi(\beta)$ is an algebra automorphism of L preserving η with $\varphi_\beta(L_\alpha) \subset L_{\beta\alpha\beta^{-1}}$ for any $\alpha \in \pi$,

(2) $\varphi_\alpha|_{L_\alpha} = \mathrm{id}_{L_\alpha}$ for any $\alpha \in \pi$,

(3) for any $a \in L_\alpha$ and $b \in L_\beta$, we have $\varphi_\beta(a)b = ba$,

(4) for any $\alpha, \beta \in \pi$ and any $c \in L_{\alpha\beta\alpha^{-1}\beta^{-1}}$, we have $\mathrm{Tr}(c\varphi_\beta \colon L_\alpha \to L_\alpha) = \mathrm{Tr}(\varphi_{\alpha^{-1}}c \colon L_\beta \to L_\beta)$, where Tr is the R-valued trace of endmorphisms of finitely generated projective R-modules; see for instance [7].

In [6], Turaev showed that there exists a relation between oriented HQFTs with target $K(\pi, 1)$ space and crossed π-algebras.

Theorem 2.9 (Turaev [6]). *Let π be a group and X be a $K(\pi, 1)$ space. Then every oriented $(1 + 1)$-dimensional HQFT with target X over the ring R determines an underlying crossed π-algebra over R. This induces a bijection between the set of isomorphism classes of oriented $(1 + 1)$-dimensional HQFTs and the set of isomorphism classes of crossed π-algebras.*

For any crossed π-algebra (L, η, φ), we denote the HQFT corresponding to the crossed π-algebra by (A^L, τ^L). Now we define extended crossed group-algebras.

Definition 2.10. Let π be a group such that $\alpha^2 = 1$ for any $\alpha \in \pi$. A tuple $(L, \eta, \varphi, \{\theta_\alpha\}_{\alpha \in \pi}, \Phi)$ is an *extended crossed π-algebra* over R if the triple (L, η, φ) is a crossed π-algebra, and the family of elements $\{\theta_\alpha \in L_1\}_{\alpha \in \pi}$ and the homomorphism of R-modules $\Phi \colon L \to L$ satisfy the following axioms:

(1) $\Phi^2 = \mathrm{id}$,

(2) $\Phi(L_\alpha) \subset L_\alpha$ for any $\alpha \in \pi$,

(3) for any $v, w \in L$, $\Phi(vw) = \Phi(w)\Phi(v)$,

(4) $\Phi(1_L) = 1_L$,

(5) $\eta \circ (\Phi \otimes \Phi) = \eta$,

(6) for any $\alpha \in \pi$, $\Phi \circ \varphi_\alpha = \varphi_\alpha \circ \Phi$,

(7) for any $\alpha, \beta, \gamma \in \pi$ and $v \in L_{\alpha\beta}$, we have

$$m \circ (\Phi \otimes \varphi_\gamma) \circ \Delta_{\alpha,\beta}(v) = \varphi_\gamma(\theta_{\alpha\gamma}\theta_\gamma v),$$
$$m \circ (\varphi_\gamma \otimes \Phi) \circ \Delta_{\alpha,\beta}(v) = \varphi_\gamma(\theta_{\beta\gamma}\theta_\gamma v),$$

where $\Delta_{\alpha,\beta} \colon L_{\alpha\beta} \to L_\alpha \otimes L_\beta$ is defined by the following relation:

(2.1) $(\mathrm{id} \otimes \eta) \circ (\Delta_{\alpha,\beta} \otimes \mathrm{id}) = m.$

Since η is non-degenerate and each L_α is finitely generated, such a map $\Delta_{\alpha,\beta}$ is uniquely determined.

(8) for any $\alpha, \beta \in \pi$ and $v \in L_\alpha$, we have $\Phi(\theta_\beta v_\alpha) = \varphi_{\beta\alpha}(\theta_{\beta\alpha} v_\alpha)$,

(9) for any $\alpha \in \pi$, we have $\Phi(\theta_\alpha) = \theta_\alpha$,

(10) for any $\alpha, \beta \in \pi$, we have $\varphi_\beta(\theta_\alpha) = \theta_\alpha$,

(11) for any $\alpha, \beta, \gamma \in \pi$, we have $\theta_\alpha\theta_\beta\theta_\gamma = q(1)\theta_{\alpha\beta\gamma}$, where $q \colon R \to L_1$ is defined as follows. Let $\{a_i \in L_{\alpha\beta}\}_{i=1}^n$ and $\{b_i \in L_{\alpha\beta}\}_{i=1}^n$ be families of elements of $L_{\alpha\beta}$ satisfying the following condition: for any $v \in L_{\alpha\beta}$

(2.2) $\sum_i \eta(b_i \otimes v)a_i = \varphi_{\beta\gamma}(v).$

From the same reason as (7), such a_i and b_i are uniquely determined. Then we put $q(1) := \sum_i a_i b_i$.

Remark 2.11. (1) Let $D_{+,+,-}(\alpha, \beta; 1, 1)$ be the oriented X-cobordism given by Figure 1. Its bottom base is a X-manifold $(\mathbf{S}^1, \alpha\beta)$ and its top base is the disjoint union of two X-manifolds $(\mathbf{S}^1, \alpha)$ and $(\mathbf{S}^1, \beta)$. Its characteristic map sends each

labeled arc to the loop corresponding to the label. Such map is uniquely determined up to homotopy since X is $K(\pi, 1)$ space. The orientation of $D_{+,+,-}(\alpha, \beta; 1, 1)$ is given by Figure 3. Then we have $\tau^L(D_{+,+,-}(\alpha, \beta; 1, 1)) = \Delta_{\alpha,\beta}$. The relation (2.1) corresponds to Figure 4.

(2) Let Q be the X-cobordism depicted in Figure 2. It is a once-punctured torus whose bottom base is empty and whose top base is a X-manifold $(\mathbf{S}^1, 1)$. Its characteristic map sends each labeled arc to the loops corresponding to the label. Such map is also uniquely determined up to homotopy. Its orientation is given by Figure 3. Then we have $q = \tau^L(Q)$. The relation (2.2) corresponds to Figure 5.
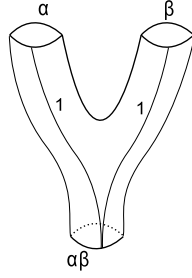
FIGURE 1. The cobordism $D_{+,+,-}(\alpha, \beta; 1, 1)$.
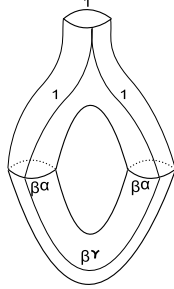
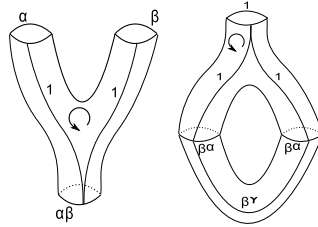FIGURE 2. Definition of the cobordism $(Q, \emptyset, (\mathbf{S}^1, 1))$.

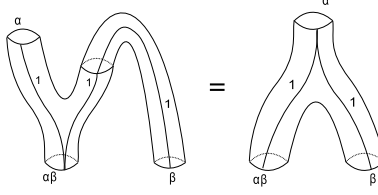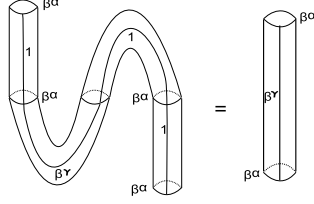
FIGURE 3. Orientations.

The following theorem is our main theorem.

FIGURE 4. Relation of cobordisms $(id \otimes \eta) \circ (\Delta_{\alpha,\beta} \otimes \mathrm{id}) = m$.

FIGURE 5. Relation of cobordisms $(\mathrm{id} \otimes \eta) \circ \tau(Q') \otimes \mathrm{id} = \varphi_{\beta\gamma}$.

Theorem 2.12 (Main theorem)**.** *Let π be a group with $\alpha^2 = 1$ for any $\alpha \in \pi$ and X be a $K(\pi, 1)$ space. Then every unoriented $(1+1)$-dimensional HQFT with target X over the ring R determines an underlying extended crossed π-algebra over R. This induces a bijection between the set of isomorphism classes of unoriented $(1+1)$-dimensional HQFTs over R and the set of isomorphism classes of extended crossed π-algebras over R.*

We have not defined "underlying extended crossed π-algebra" yet. We will define it in Section 3.

3. Underlying algebraic structures of HQFTs

In this section, we construct an extended crossed group algebra from an HQFT. Assume that π is a group such that any element $\alpha \in \pi$ satisfies $\alpha^2 = 1(= 1_\pi)$, where 1_π is the unit of π (in particular, π is an abelian group). Moreover let X be a $K(\pi, 1)$ space with a base point $x_0 \in X$. Throughout this section, let (A, τ) be an unoriented $(1 + 1)$-dimensional HQFT with target X. Let $\mathbf{S}^1$ be an (unoriented) circle. For any unoriented 1-dimensional X-manifold $(\mathbf{S}^1, g)$, if we give $\mathbf{S}^1$ an orientation, we can regard the homotopy class of g as an element $\alpha \in \pi = \pi_1(X)$. The element α does not depend on the choice of the orientation of $\mathbf{S}^1$ since $\alpha = \alpha^{-1}$. Since we consider the module $A(\mathbf{S}^1, g)$, we can denote the unoriented 1-dimensional X-manifold $(\mathbf{S}^1, g)$ by $(\mathbf{S}^1, \alpha)$.

Definition 3.1. Let Mb be a Möbius band. For any $\alpha \in \pi$, we define an unoriented $(1+1)$-dimensional X-cobordism $(Mb, \emptyset, \partial(Mb), g_\alpha)$ as the unoriented $(1+1)$-dimensional X-cobordism in Figure 6. The map g_α is a continuous map from Mb to X and is uniquely determined by the element α up to homotopy since X is a $K(\pi, 1)$ space. Choose an unoriented X-homeomorphism $f \colon (\partial(Mb), g|_{\partial(Mb)}) \to (\mathbf{S}^1, 1)$, and define an element θ_α by

$$\theta_\alpha := f_\sharp(\tau((Mb, \emptyset, \partial(Mb), g_\alpha))(1)) \in A(\mathbf{S}^1, 1).$$
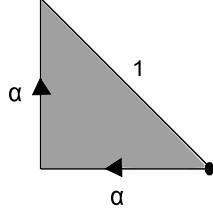
FIGURE 6. The cobordism $(Mb, \emptyset, \partial(Mb), g_\alpha)$. Three edges are labeled by $\alpha, \alpha, 1 \in \pi$. Two edges of the triangle labeled by $\alpha \in \pi$ are identified along the arrows depicted in Figure 6. These labels mean that the map g sends each edge to the loop corresponding to the label.

Lemma 3.2. *The element θ_α does not depend on the choice of an unoriented X-homeomorphism f.*

Proof. Let f and $f' \colon (\partial(Mb), g|_{\partial(Mb)}) \to (\mathbf{S}^1, 1)$ be unoriented X-homeomorphisms such that f is not isotopic to f'. Let $T \colon Mb \to Mb$ be a homeomorphism reversing the orientation of the boundary. The map T induces an unoriented X-homeomorphism $T \colon (Mb, g_\alpha) \to (Mb, g_\alpha)$. Then $f \circ T|_{\partial(Mb)}$ is isotopic to f'. By Definition 2.5, we have

$$
\begin{aligned}
f_\sharp(\tau(Mb, g_\alpha)(1)) &= f_\sharp(\tau(T(Mb), g_\alpha)(1)) \\
&= f_\sharp \circ (T|_{\partial(Mb)})(\tau(Mb, g_\alpha)(1)) \\
&= f'_\sharp(\tau(Mb, g_\alpha)(1)).
\end{aligned}
$$

Therefore the element θ_α does not depend on the choice of f. □

Let $\chi \colon \mathbf{S}^1 \to \mathbf{S}^1$ be a homeomorphism reversing the orientation. For any $\alpha \in \pi$, we define an isomorphism of R-modules $\Phi_\alpha = \Phi \colon A(\mathbf{S}^1, \alpha) \to A(\mathbf{S}^1, \alpha)$ by $\chi_\sharp \colon A(\mathbf{S}^1, \alpha) \to A(\mathbf{S}^1, \alpha)$, where $\chi_\sharp$ is the R-homomorphism induced by χ. Clearly we have $\Phi(\theta_\alpha) = \theta_\alpha$ for any $\alpha \in \pi$.
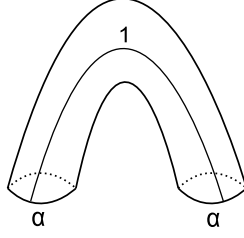
For any $\alpha \in \pi$, let $C_{-,-}(\alpha; 1)$ be an unoriented X-cobordism depicted in Figure 7. The unoriented X-cobordism $C_{-,-}(\alpha; 1)$ is an annulus whose bottom base is the disjoint union of two copies of X-manifolds $(\mathbf{S}^1, \alpha)$, whose top base is empty and whose characteristic map sends the arc labeled by $1 \in \pi$ onto $x_0 \in X$. See Figure 7. Such a map is also uniquely determined up to homotopy by the element $\alpha \in \pi$. For any element $\alpha \in \pi$, we define a homomorphism of R-modules $\eta_\alpha = \eta \colon A(\mathbf{S}^1, \alpha) \otimes A(\mathbf{S}^1, \alpha) \to R$ by $\tau(C_{-,-}(\alpha, 1))$.

Lemma 3.3. *We have $\eta \circ (\Phi \otimes \Phi) = \eta$.*

*Proof.* Let a map $\mu \colon C_{-,-}(\alpha; 1) \to C_{-,-}(\alpha; 1)$ be an orientation reversing homeomorphism. From Definition 2.5, we have

$$
\tau(C_{-,-}(\alpha, 1)) = \tau(\mu(C_{-,-}(\alpha, 1))) \circ (\mu|_{(\mathbf{S}^1, \alpha) \sqcup (\mathbf{S}^1, \alpha)})_\sharp = \tau(C_{-,-}(\alpha, 1)) \circ (\Phi \otimes \Phi).
$$

□

FIGURE 7. The cobordism $C_{-,-}(\alpha; 1)$.

For any $\alpha, \beta \in \pi$, let $D_{-,-,+}(\alpha, \beta; 1, 1)$ be an unoriented X-cobordism depicted in Figure 8. The unoriented X-cobordism $D_{-,-,+}(\alpha, \beta; 1, 1)$ is a twice-punctured disk whose bottom base is the disjoint union of two unoriented X-manifolds $(\mathbf{S}^1, \alpha)$ and $(\mathbf{S}^1, \beta)$, whose top base is an unoriented X-manifold $(\mathbf{S}^1, \alpha\beta)$ and whose characteristic map sends the arcs labeled by $1 \in \pi$ onto $x_0 \in X$. See Figure 8. Such a map is also uniquely determined by the elements $\alpha, \beta \in \pi$ up to homotopy. For any $\alpha, \beta \in \pi$, we define a homomorphism of R-modules $m = m_{\alpha,\beta} \colon A(\mathbf{S}^1, \alpha) \otimes A(\mathbf{S}^1, \beta) \to A(\mathbf{S}^1, \alpha\beta)$ by $\tau(D_{-,-,+}(\alpha, \beta; 1, 1))$. For any $v \in A(\mathbf{S}^1, \alpha)$ and $w \in A(\mathbf{S}^1, \beta)$, we denote $m(v \otimes w)$ by $vw \in A(\mathbf{S}^1, \alpha\beta)$.
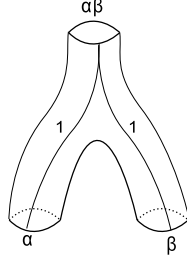
FIGURE 8. The cobordism $D_{-,-,+}(\alpha, \beta; 1, 1)$.

Lemma 3.4. *For two elements $v \in A(\mathbf{S}^1, \alpha)$ and $w \in A(\mathbf{S}^1, \beta)$, we have $\Phi(vw) = \Phi(w)\Phi(v)$.*

Proof. The proof of this lemma is similar to that of Lemma 3.3. ☐

For any $\alpha, \beta \in \pi$, let $C_{-,+}(\alpha; \beta)$ be an unoriented X-cobordism depicted in Figure 9. The unoriented X-cobordism $C_{-,+}(\alpha; \beta)$ is an annulus whose bottom base is an unoriented X-manifold $(\mathbf{S}^1, \alpha)$, whose top base is also an unoriented X-manifold $(\mathbf{S}^1, \alpha)$ and whose characteristic map sends the arc labeled by $\beta \in \pi$ onto a loop on X whose homotopy class is $\beta \in \pi$. See Figure 9. Such a map is also uniquely determined up to homotopy by the elements $\alpha, \beta \in \pi$. For any $\alpha, \beta \in \pi$, we define a homomorphism of R-modules $\varphi_\beta \colon \bigoplus_{\alpha \in \pi} A(\mathbf{S}^1, \alpha) \to \bigoplus_{\alpha \in \pi} A(\mathbf{S}^1, \alpha)$ by $\bigoplus_{\alpha \in \pi} \tau(C_{-,+}(\alpha; \beta))$.

Lemma 3.5. *For any $\beta \in \pi$, we have $\Phi \circ \varphi_\beta \circ \Phi = \varphi_\beta$.*

Proof. We can prove this lemma by using the same argument as Lemma 3.3. ☐

FIGURE 9. The cobordism $C_{-,+}(\alpha;\beta)$.

Lemma 3.6. *For any $\alpha \in \pi$ and $v_\alpha \in A(\mathbf{S}^1, \alpha)$, we have $\Phi(\theta_\beta v_\alpha) = \varphi_{\beta\alpha}(\theta_{\beta\alpha}v_\alpha)$.*

Proof. Figure 10 shows this lemma. In Figure 10 the first cobordism corresponds to $\theta_\beta v_\alpha$ and the fifth cobordism corresponds to $\varphi_{\beta\alpha}(\theta_{\beta\alpha}v_\alpha)$, where two arrows depicted in Figure 10 mean that two edges endowed with the arrows are identified respecting the orientations indicated by them. Sliding the top base of the first cobordism, we get the second, the third and the fourth cobordisms. As a result the top base is reversed. From these transformations and Definition 2.5 we have $\Phi(\theta_\beta v_\alpha) = \varphi_{\beta\alpha}(\theta_{\beta\alpha}v_\alpha)$. $\qquad\square$

Lemma 3.7. *For any $\alpha, \beta \in \pi$, we have $\varphi_\beta(\theta_\alpha) = \theta_\alpha$.*

Proof. See Figure 11. In Figure 11 the first cobordism corresponds to $\varphi_\beta(\theta_\alpha)$, where arrows depicted in Figure 11 mean that edges endowed with these arrows are identified along the same arrows. The fourth cobordism corresponds to θ_α because π is an abelian group and any element $\alpha \in \pi$ satisfies $\alpha^2 = 1$. $\qquad\square$

Lemma 3.8. *For any $\alpha, \beta, \gamma \in \pi$, let Q be the unoriented $(1+1)$-dimensional X-cobordism introduced in Definition 2.10 and depicted in Figure 2. Then we have $\theta_\alpha \theta_\beta \theta_\gamma = \tau(Q)(1)\theta_{\alpha\beta\gamma}$, where 1 is the unit of R.*

Proof. Figure 12 shows this lemma. In Figure 12 the first cobordism corresponds to $\theta_\alpha \theta_\beta \theta_\gamma$ and the eighth cobordism corresponds to $\tau(Q)(1)\theta_{\alpha\beta\gamma}$. $\qquad\square$

Lemma 3.9. *For any $\alpha, \beta, \gamma \in \pi$ and $v \in A(\mathbf{S}^1, \alpha\beta)$, we have the following equations:*

$$m \circ (\Phi \otimes \varphi_\gamma) \circ \Delta_{\alpha,\beta}(v) = \varphi_\gamma(\theta_{\alpha\gamma}\theta_\gamma v),$$
$$m \circ (\varphi_\gamma \otimes \Phi) \circ \Delta_{\alpha,\beta}(v) = \varphi_\gamma(\theta_{\beta\gamma}\theta_\gamma v),$$

where $\Delta_{\alpha,\beta} = \tau(D_{+,+,-}(\alpha,\beta;1,1))$ and $D_{+,+,-}(\alpha,\beta;1,1)$ is the unoriented X-cobordism introduced in Definition 2.10.

Proof. This lemma follows from Figure 13. In Figure 13 the first cobordism corresponds to $m \circ (\varphi_\gamma \otimes \Phi) \circ \Delta_{\alpha,\beta}(v)$ and the fourth cobordism corresponds to $\varphi_\gamma(\theta_{\alpha\gamma}\theta_\gamma v)$. Similarly we can prove $m \circ (\varphi_\gamma \otimes \Phi) \circ \Delta_{\alpha,\beta}(v) = \varphi_\gamma(\theta_{\beta\gamma}\theta_\gamma v)$. $\qquad\square$

Lemma 3.10. *We have $\Phi(1_L) = 1_L$, where $1_L = \tau(D, \emptyset, \partial D)(1)$, 1 is the unit of R and D is a cup which is an unoriented X-cobordism depicted in Figure 14. Note that the characteristic map of D is uniquely determined.*

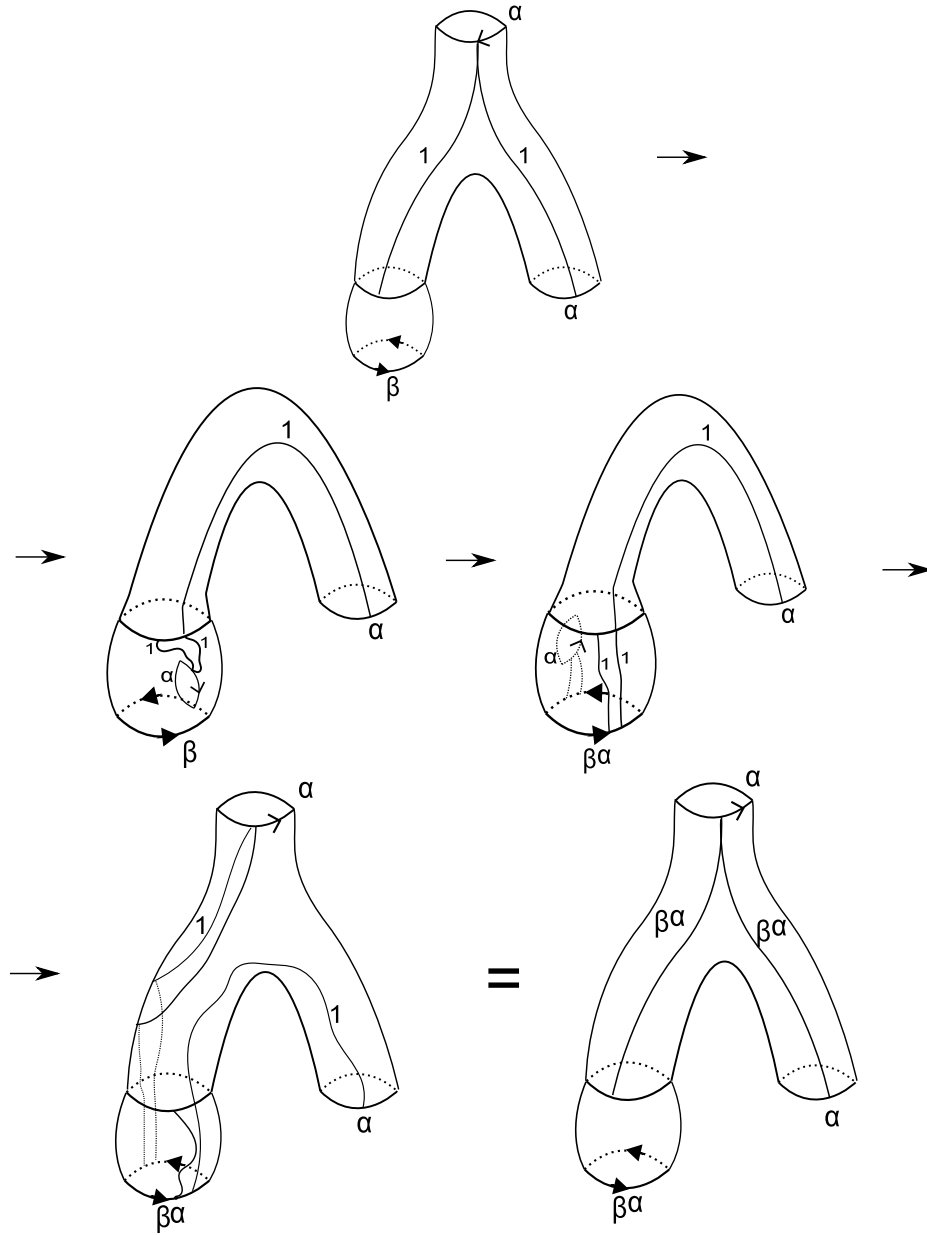*Proof.* By using similar argument of Lemma 3.3 we can prove this. $\qquad\square$

FIGURE 10. Proof of Lemma 3.6.

Theorem 2.9 and Lemmas 3.2-3.10 show that any unoriented $(1+1)$-dimensional HQFT (A, τ) with target X induces an extended crossed π-algebra. We call the extended crossed π-algebra the underlying extended crossed π-algebra of the unoriented $(1+1)$-dimensional HQFT (A, τ).
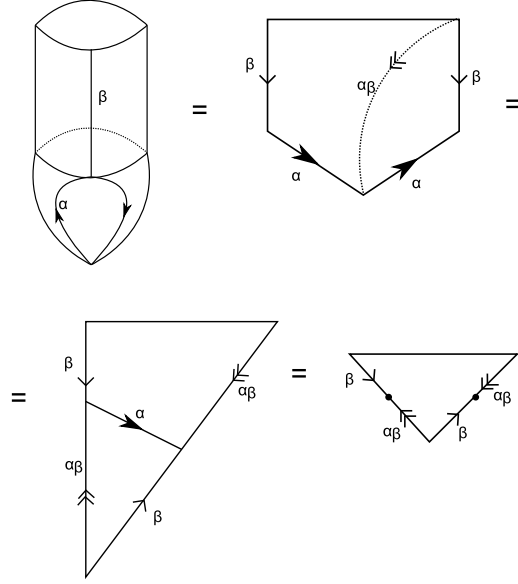
FIGURE 11. Proof of Lemma 3.7.

4. PROOF OF MAIN THEOREM

In this section, we prove Theorem 2.12. To prove the theorem, we need to make an unoriented HQFT (A, τ) from a given extended crossed π-algebra $(L, \eta, \varphi, \{\theta_\alpha\}_{\alpha \in \pi}, \Phi)$.

Our proof has three steps. In Step 1, we construct a functor A. In Step 2, we make a functor τ. In Step 3, we prove that the pair (A, τ) satisfies the axioms of HQFTs. To construct them we use the same method as [8].

Step 1: Construction of a functor A.

Let (M, g) be a connected unoriented 1-dimensional X-manifold. We define an R-module $A(M, g)$ by

$$A(M, g)$$
$$:= \{(r, v) | r : (\mathbf{S}^1, \alpha) \to (M, g) : \text{unoriented } X-\text{homeomorphism}, v \in L_\alpha\} / \approx,$$

where $(r, v) \approx (r,' v')$ if and only if r is isotopic to r' and $v = v'$, or r is not isotopic to r' and $v = \Phi(v')$. For any (M, g), such $\alpha \in \pi$ is uniquely determined. For any unoriented X-homeomorphism $h : (\mathbf{S}^1, \alpha) \to (M, g)$, we define a map $\widetilde{h} : A(M, g) \to L_\alpha$ by

$$\widetilde{h}(r, v) := \begin{cases} v & (\text{if } r \text{ is isotopic to } h), \\ \Phi(v) & (\text{if } r \text{ is not isotopic to } h). \end{cases}$$

Then the map $\widetilde{h}$ is bijective. In fact it has inverse map $(\widetilde{h})^{-1} : L_\alpha \to A(M, g)$ which is defined by $(\widetilde{h})^{-1}(v) = (h, v)$ for any $v \in L_\alpha$. Moreover we can use the R-module structure of L_α to turn $A(M, g)$ into an R-module. The R-module structure of
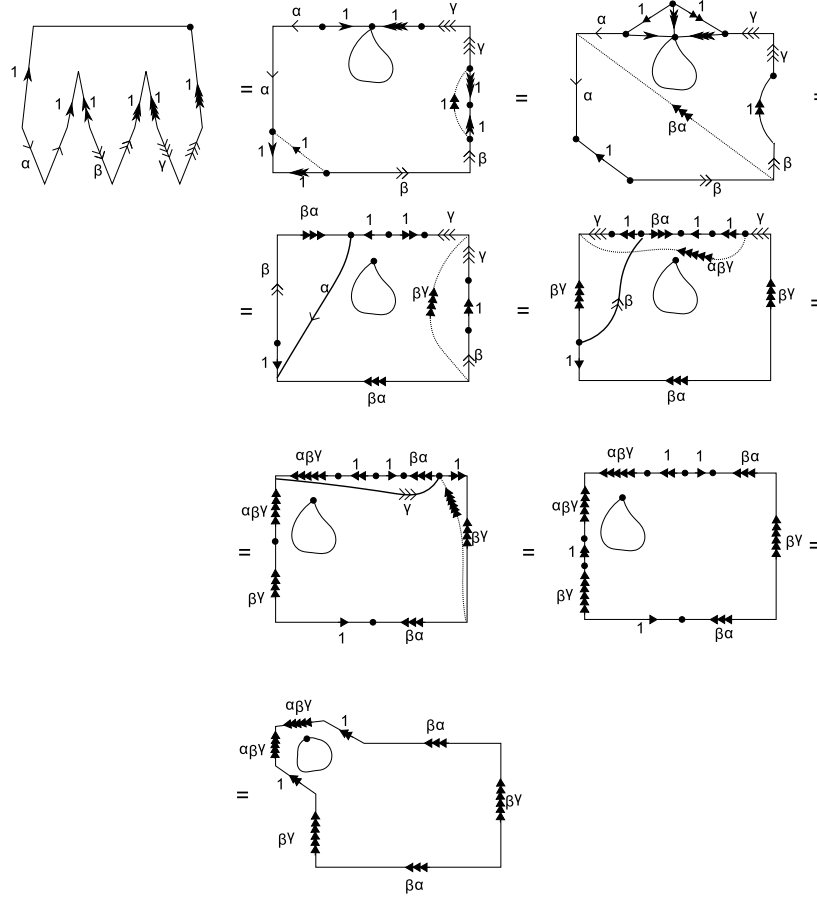
FIGURE 12. Proof of Lemma 3.8. As usual, we identify edges with some types of arrows in these pictures.

$A(M, g)$ does not depend on the choice of the map h. This follows from the folowing:

$$
\begin{aligned}
a(r, v) &= (\widetilde{h})^{-1}(a\widetilde{h}(r, v)) \\
&= \begin{cases} (h, av) & (\text{if } r \text{ is isotopic to } h), \\ (h, a\Phi(v)) & (\text{if } r \text{ is not isotopic to } h) \end{cases} \\
&= \begin{cases} (r, av) & (\text{if } r \text{ is isotopic to } h), \\ (r, \Phi(a\Phi(v))) & (\text{if } r \text{ is not isotopic to } h) \end{cases} \\
&= (r, av),
\end{aligned}
$$

where $(r, v) \in A(M, g)$ and $a \in R$. Since L_α is a projective R-module, so is $A(M, g)$. In general we define $A(\emptyset)$ by R and $A(M \sqcup N)$ by $A(M) \otimes A(N)$ for all connected unoriented 1-dimensional X-manifolds M and N (more precisely $M \sqcup N$ is an ordered disjoint union and $A(M) \otimes A(N)$ is an ordered tensor product). For any unoriented X-homeomorphism of unoriented X-manifolds $f \colon (M, g) \to (M', g')$, we define an R-homomorphism $f_\sharp \colon A(M, g) \to A(M', g')$ by $f_\sharp(r, v) = (f \circ r, v)$ for any
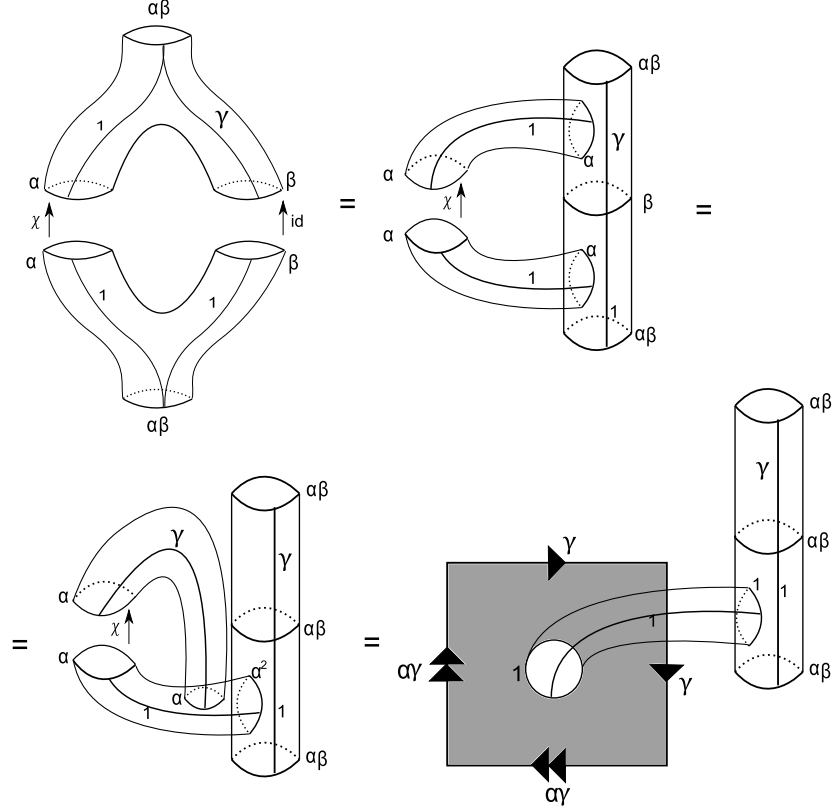
FIGURE 13. Proof of Lemma 3.9. $m \circ (\Phi \otimes \varphi_\gamma) \circ \Delta_{\alpha,\beta}(v) = \varphi_\gamma(\theta_{\alpha\gamma}\theta_\gamma v)$. In the last picture, we identify edges with some types of arrows in these pictures.

FIGURE 14. The cobordism $(D, \emptyset, \partial D)$.

$(r, v) \in A(M, g)$. Clearly $f_\sharp$ is an R-isomorphism and preserved under isotopies of f.

Step 2: Construction of a functor τ.

For any unoriented $(1 + 1)$-dimensional X-cobordism (W, M_0, M_1, g), we define an R-homomorphism $\tau(W, g) \colon A(M_0, g|_{M_0}) \to A(M_1, g|_{M_1})$ as follows:

Case 1: W is orientable and connected.

Fix an orientation of $\mathbf{S}^1$ and give W an orientation, and take unoriented X-homeomorphisms $h_{M_0} \colon (\mathbf{S}^1, \alpha_1) \sqcup \cdots \sqcup (\mathbf{S}^1, \alpha_n) \to (M_0, g|_{M_0})$ and $h_{M_1} \colon (\mathbf{S}^1, \beta_1) \sqcup \cdots \sqcup (\mathbf{S}^1, \beta_n) \to (M_1, g|_{M_1})$ which preserve orientations. Then we define an R-homomorphism $\tau(W, g) \colon A(M_0, g|_{M_0}) \to A(M_1, g|_{M_1})$ by $\widetilde{h_{M_1}}^{-1} \circ \tau^L(W, g) \circ \widetilde{h_{M_0}}$. The definition of τ^L is introduced in Theorem 2.9. We need to prove that $\tau(W, g)$

does not depend on the choice of their orientations. It is sufficient that we check it in the cases where W is an unoriented basic cobordism depicted in Figure 15. When W is an unorieted X-cobordism at the upper left in Figure 15, take unoriented X-homeomorphisms $h_{M_0} \colon (\mathbf{S}^1, \alpha) \sqcup (\mathbf{S}^1, \beta) \to M_0$ and $h_{M_1} \colon (\mathbf{S}^1, \alpha\beta) \to M_1$. Then we have

$$\widetilde{h_{M_1}}^{-1} \circ m \circ \widetilde{h_{M_0}} = \widetilde{h_{M_1}}^{-1} \circ \Phi \circ m \circ P \circ (\Phi \otimes \Phi) \circ \widetilde{h_{M_0}}$$

$$= (\widetilde{h_{M_1} \circ \chi})^{-1} \circ m \circ P \circ (\widetilde{h_{M_0} \circ (\chi \sqcup \chi)}),$$

where P is the permutation. This equation implies that $\tau(W, g)$ does not depend on the choice of the orientation of W. In other cases, we can use similar arguments.
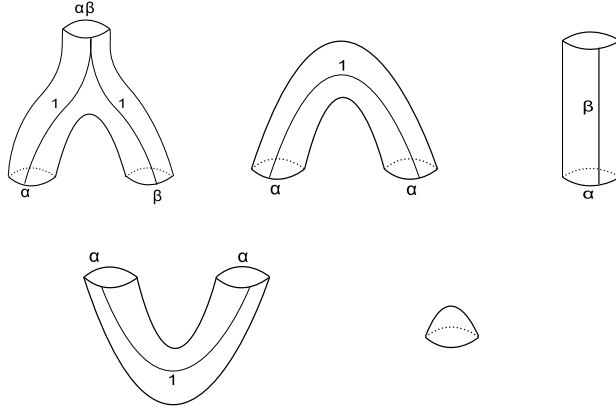

FIGURE 15. Oriented basic cobordisms.

Case 2: W is non-orientable and connected.

Let $\mathbf{R}P^2$ be the projective plane. For any $\alpha \in \pi$, we define an unoriented X-cobordism $(\mathbf{R}P^2, f_\alpha, p)$ with p a point of $\mathbf{R}P^2$ as follows. The pair $(\mathbf{R}P^2, f_\alpha)$ is an unorieted X-cobordism $(\mathbf{R}P^2, \emptyset, \emptyset, f_\alpha)$ such that $f_\alpha(p) = x_0$ and that the homotopy class of $f_\alpha|_l$ equals $\alpha \in \pi$ for the loop l on $\mathbf{R}P^2$ depicted in Figure 16 (in Figure 16, l is presented by the upper arc with arrow, which is identified with the lower arc with arrow). Such unoriented X-cobordism is uniquely determined up to homotopy by $\alpha := [l] \in \pi$. In general for any unoriented X-cobordism $(\mathbf{R}P^2, g)$, (by using
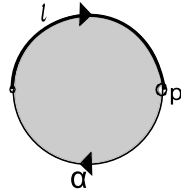

FIGURE 16. The cobordism $(\mathbf{R}P^2, f_\alpha)$D

homotopy) we can assume that there are $p \in \mathbf{R}P^2$ and $\alpha \in \pi$ which satisfy the following conditions:

- $g(p) = x_0$,
- g is homotopic to f_α.

We denote the unoriented X-cobordism $(\mathbf{R}P^2, g)$ by $(\mathbf{R}P^2, f_\alpha)$. Now we represent (W, g) as the connected sum of an orientable (unoriented) X-cobordism $(W^{\mathrm{or}}, g|_{W^{\mathrm{or}}})$ and unoriented X-cobordisms $(\mathbf{R}P^2, f_{\alpha_1}), \ldots, (\mathbf{R}P^2, f_{\alpha_n})$, that is,

$$(W, g) \cong (W^{\mathrm{or}}, g|_{W^{\mathrm{or}}}) \sharp (\mathbf{R}P^2, f_{\alpha_1}) \sharp \ldots \sharp (\mathbf{R}P^2, f_{\alpha_n}).$$

Note that $\partial W^{\mathrm{or}} = \partial W$ and that a homomorphism $\tau(W^{\mathrm{or}}, g|_{W^{\mathrm{or}}})$ is defined in the orientable case. Let m be the number of components of M_1. We define a homomorphism $\tau(W, g)$ as follows. If $m > 0$, take an unoriented X-homeomorphism $h \colon \bigsqcup_i^m (\mathbf{S}^1, \beta_i) \to (M_1, g|_{M_1})$ and identify two R-modules $A(M_1, g|_{M_1})$ and $\bigotimes_{i=1}^m L_{\beta_i}$ by $\tilde{h} \colon A(M_1, g|_{M_1}) \to \bigotimes_{i=1}^m L_{\beta_i}$. Under this identification, we define a map $\psi_{\alpha_1, \cdots, \alpha_n} \colon A(M_1, g|_{M_1}) \to A(M_1, g|_{M_1})$ to be the identity on all factors except one where it is multiplication by $\prod_{i=1}^n \theta_{\alpha_i}$. We define

$$\tau(W, g) := \psi_{\alpha_1, \cdots, \alpha_n} \circ \tau(W^{\mathrm{or}}, g|_{W^{\mathrm{or}}}).$$

If $m = 0$, consider an unoriented X-cobordism $W^{\mathrm{or}} - D^2 = (W^{\mathrm{or}} - D^2, M_0, M_1 \sqcup \partial(D^2), g|_{W^{\mathrm{or}} - D^2})$, where D^2 is any disk on W. Take an unoriented X-homeomorphism $h \colon (\mathbf{S}^1, 1) \to (\partial(D^2), g|_{\partial(D^2)})$ and identify two R-modules $A(\partial(D^2), g|_{\partial(D^2)})$ and L_1 by $\tilde{h} \colon A(\partial(D^2), g|_{\partial(D^2)}) \to L_1$. Under this identification, we define a homomorphism $\tau(W, g)$ by

$$\tau(W, g)(v) := \eta(\tau(W^{\mathrm{or}} - D^2, g|_{W^{\mathrm{or}} - D^2})(v), \prod_{i=1}^n \theta_{\alpha_i})$$

for any $v \in A(M_0, g|_{M_0})$. From Lemmas 4.1 and 4.2 below, the functor τ is well defined.

Case 3: W is not connected:

We can extend the definition of τ constructed as above to non-connected cases by using tensor products as Step1.

Step 3: The pair (A, τ) is an unoriented $(1 + 1)$-dimensional HQFT with target X.

We need to check the axioms of unoriented HQFTs (see Definition 2.5). The pair (A, τ) clearly satisfies the axioms except for (4) and (6). In the next section, we show that (A, τ) satisfies the axioms (4) and (6) (Propositions 5.1 and 5.4).

From Step 1, 2 and 3, we complete the proof of Theorem 2.12 (except for Propositions 5.1 and 5.4 and Lemmas 4.1 and 4.2).

Lemma 4.1. (i) *The map $\tau(W, g)$ does not depend on the choice of h.*

(ii) *The map $\tau(W, g)$ does not depend on the choice of a factor multiplied the element $\prod_{i=1}^n \theta_{\alpha_i} \in L_1$.*

(iii) *The map $\tau(W, g)$ does not depend on the choice of the connected sum $(W, g) \cong (W^{\mathrm{or}}, g|_{W^{\mathrm{or}}}) \sharp (\mathbf{R}P^2, f_{\alpha_1}) \sharp \ldots \sharp (\mathbf{R}P^2, f_{\alpha_n})$.*

Proof. (i): In the case where $m = 1$, take any unoriented X-homeomorphism $h\colon (\mathbf{S}^1, \alpha) \to (M_1, g|_{M_1})$. For any $(r, v) \in A(M_1, g|_{M_1})$, we have

$$(\tilde{h})^{-1}(\prod_{i=1}^{n} \theta_{\alpha_i} \tilde{h}(r, v)) = \begin{cases} (h, \prod_{i=1}^{n} \theta_{\alpha_i} v) & \text{(if } r \text{ is isotopic to } h\text{)}, \\ (h, \prod_{i=1}^{n} \theta_{\alpha_i} \Phi(v)) & \text{(if } r \text{ is not isotopic to } h\text{)} \end{cases}$$

$$= \begin{cases} (r, \prod_{i=1}^{n} \theta_{\alpha_i} v) & \text{(if } r \text{ is isotopic to } h\text{)}, \\ (r, \Phi(\prod_{i=1}^{n} \theta_{\alpha_i} \Phi(v))) & \text{(if } r \text{ is not isotopic to } h\text{)} \end{cases}$$

$$= (r, \prod_{i=1}^{n} \theta_{\alpha_i} v).$$

Hence the map $\tau(W, g)$ does not depend on the choice of h. Similarly we can prove the case where $m > 1$. In the case where $m = 0$, (i) follows from the fact that Φ preserves η and θ_α for any α.

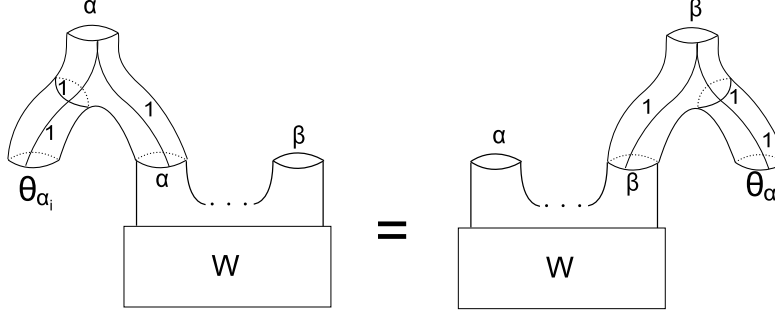(ii): It follows from Figure 17 and Theorem 2.9.


FIGURE 17. Cobordism relation.

(iii): In the proof of Lemma 3.8, we proved that $(\mathbf{R}P^2, f_\alpha) \sharp (\mathbf{R}P^2, f_\beta) \sharp (\mathbf{R}P^2, f_\gamma)$ is unoriented X-homeomorphic to $(T^2, g_{\alpha,\beta,\gamma}) \sharp (\mathbf{R}P^2, f_{\alpha\beta\gamma})$, where $(T^2, g_{\alpha,\beta,\gamma})$ is the unoriented X-cobordism depicted in Figure 18 whose bottom base and top base are empty and whose characteristic map $g_{\alpha,\beta,\gamma}$ sends the arcs labeled by $1, \beta\alpha, \beta\gamma \in \pi$ onto the loops with the corresponding labels. Such characteristic map is uniquely determined by these labels. It follows from the definitions that $\tau((\mathbf{R}P^2, f_\alpha) \sharp (\mathbf{R}P^2, f_\beta) \sharp (\mathbf{R}P^2, f_\gamma)) = \tau((T^2, g_{\alpha,\beta,\gamma}) \sharp (\mathbf{R}P^2, f_{\alpha\beta\gamma}))$. Hence it is sufficient to prove (iii) for the case where $n = 1$ or 2, which is shown in Lemma 4.2 below. $\square$

Lemma 4.2. (I) *Assume that we have two connected sums*

$$(W, g) \cong (W^{\mathrm{or}}, g|_{W^{\mathrm{or}}}) \sharp (\mathbf{R}P^2, f_\alpha)$$

and

$$(W, g) \cong (\widetilde{W}^{\mathrm{or}}, g|_{\widetilde{W}^{\mathrm{or}}}) \sharp (\mathbf{R}P^2, f_\beta).$$
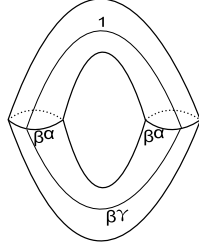
FIGURE 18. The cobordism $(T^2, g_{\alpha,\beta,\gamma})$D

(I-a) *If we have an unoriented X-homeomorphism of unoriented X-manifolds* $f\colon (W^{\mathrm{or}}, g|_{W^{\mathrm{or}}}) \to (\widetilde{W}^{\mathrm{or}}, g|_{\widetilde{W}^{\mathrm{or}}})$ *and* $\alpha = \beta$, *we have* $\psi_\alpha \circ \tau(W^{\mathrm{or}}, g|_{W^{\mathrm{or}}}) = \psi_\beta \circ \tau(\widetilde{W}^{\mathrm{or}}, g|_{\widetilde{W}^{\mathrm{or}}})$.

(I-b) *If we have an unoriented X-homeomorphism of unoriented X-manifolds* $f\colon (\mathbf{R}P^2, f_\alpha) \to (\mathbf{R}P^2, f_\beta)$ *and* $W^{\mathrm{or}} = \widetilde{W}^{\mathrm{or}}$, *we have* $\psi_\alpha \circ \tau(W^{\mathrm{or}}, g|_{W^{\mathrm{or}}}) = \psi_\beta \circ \tau(\widetilde{W}^{\mathrm{or}}, g|_{\widetilde{W}^{\mathrm{or}}})$.

(I-c) *We have* $\psi_\alpha \circ \tau(W^{\mathrm{or}}, g|_{W^{\mathrm{or}}}) = \psi_\beta \circ \tau(\widetilde{W}^{\mathrm{or}}, g|_{\widetilde{W}^{\mathrm{or}}})$.

(II) *Assume that we have two connected sums*

$$(W, g) \cong (W^{\mathrm{or}}, g|_{W^{\mathrm{or}}}) \sharp (\mathbf{R}P^2, f_{\alpha_1}) \sharp (\mathbf{R}P^2, f_{\alpha_2})$$

and

$$(W, g) \cong (\widetilde{W}^{\mathrm{or}}, g|_{\widetilde{W}^{\mathrm{or}}}) \sharp (\mathbf{R}P^2, f_{\beta_1}) \sharp (\mathbf{R}P^2, f_{\beta_2}).$$

(II-a) *If we have an unoriented X-homeomorphism of unoriented X-manifolds* $f\colon (W^{\mathrm{or}}, g|_{W^{\mathrm{or}}}) \to (\widetilde{W}^{\mathrm{or}}, g|_{\widetilde{W}^{\mathrm{or}}})$ *and* $\{\alpha_1, \alpha_2\} = \{\beta_1, \beta_2\}$, *we have* $\psi_{\alpha_1,\alpha_2} \circ \tau(W^{\mathrm{or}}, g|_{W^{\mathrm{or}}}) = \psi_{\beta_1,\beta_2} \circ \tau(\widetilde{W}^{\mathrm{or}}, g|_{\widetilde{W}^{\mathrm{or}}})$.

(II-b) *If we have an unoriented X-homeomorphism of unoriented X-manifolds* $f\colon (\mathbf{R}P^2, f_{\alpha_1}) \sharp (\mathbf{R}P^2, f_{\alpha_2}) \to (\mathbf{R}P^2, f_{\beta_1}) \sharp (\mathbf{R}P^2, f_{\beta_2})$ *and* $W^{\mathrm{or}} = \widetilde{W}^{\mathrm{or}}$, *we have* $\psi_{\alpha_1,\alpha_2} \circ \tau(W^{\mathrm{or}}, g|_{W^{\mathrm{or}}}) = \psi_{\beta_1,\beta_2} \circ \tau(\widetilde{W}^{\mathrm{or}}, g|_{\widetilde{W}^{\mathrm{or}}})$.

(II-c)*For any such connected sum, we have* $\psi_{\alpha_1,\alpha_2} \circ \tau(W^{\mathrm{or}}, g|_{W^{\mathrm{or}}}) = \psi_{\beta_1,\beta_2} \circ \tau(\widetilde{W}^{\mathrm{or}}, g|_{\widetilde{W}^{\mathrm{or}}})$.

Proof. (I-a): We can naturally identify $\partial(W^{\mathrm{or}})$ and $\partial(\widetilde{W}^{\mathrm{or}})$ with $\partial(W)$. It follows from the definition of $\tau(W^{\mathrm{or}})$ and $\tau(\widetilde{W}^{\mathrm{or}})$ and Theorem 2.5 that

$$(4.1) \qquad (f|_{M_1})_\sharp \circ \tau(W^{\mathrm{or}}) = \tau(\widetilde{W}^{\mathrm{or}}) \circ (f|_{M_0})_\sharp.$$

The map $(f|_{M_1})_\sharp$ is the identity map or Φ on each factor of $A(M_1, g|_{M_1})$. It follows from the definition of extended crossed π-algebras that the center of L contains L_1 and that $\Phi(\theta_\alpha v) = \theta_\alpha \Phi(v)$ for any $\alpha \in \pi$, $v \in L$. Hence we have

$$(4.2) \qquad \psi_\alpha \circ (f|_{M_1})_\sharp = (f|_{M_1})_\sharp \circ \psi_\alpha.$$

It follows from (4.1) and (4.2) that

$$(4.3) \qquad (f|_{M_1})_\sharp \circ \psi_\alpha \circ \tau(W^{\mathrm{or}}) = \psi_\alpha \circ \tau(\widetilde{W}^{\mathrm{or}}) \circ (f|_{M_0})_\sharp.$$

The equation (4.3) implies (I-a).

(I-b): Since the mapping class group of the projective plane $\mathbf{R}P^2$ is trivial, the map f is isotopic to the identity map, that is, there exists a continuous map

$H \colon \mathbf{R}P^2 \times [0,1] \to \mathbf{R}P^2$ such that the map $H_t \colon \mathbf{R}P^2 \to \mathbf{R}P^2$ defined by $H_t(x) = H(x,t)$ for any $x \in \mathbf{R}P^2$ is a homeomorphism for all $t \in [0,1]$ and satisfies $H_0 = f$ and $H_1 = \mathrm{id}$. We do not know if the map H fixes $p \in \mathbf{R}P^2$, where p is the point introduced in the proof of Theorem 2.12. Let $\gamma \in \pi$ be an element corresponding to the loop $f_\beta(H(p \otimes [0,1]))$ on X. Then we have $\alpha = \gamma\beta\gamma = \beta\gamma^2 = \beta$. This implies (I-b).

(I-c): Consider the loops c, $f(c)$ and c' as depicted in Figure 19. In general, $f(c)$ is not homotopic to c'. For any $\alpha, \beta \in \pi$ and $v \in L_\alpha$, we have $\Phi(\theta_\beta v) = \varphi_{\alpha\beta}(\theta_{\alpha\beta} v)$. This equation means that $\tau(W, g)$ is preserved under the transformation depicted in Figure 20 (see the proof of Lemma 3.6). By the definition of $\tau(W, g)$ and Theorem 2.9, the map $\tau(W, g)$ is preserved by Dehn twists on W. By using the transformation depicted in Figure 20 and Dehn twists, we can assume that $f(c)$ is homotopic to c'. If $f(c)$ is homotopic to c', there exists an unoriented X-homeomorphism $f' \colon (\mathbf{R}P^2, f_\alpha) \to (\mathbf{R}P^2, f_\beta)$ and we can use arguments in (I-b).
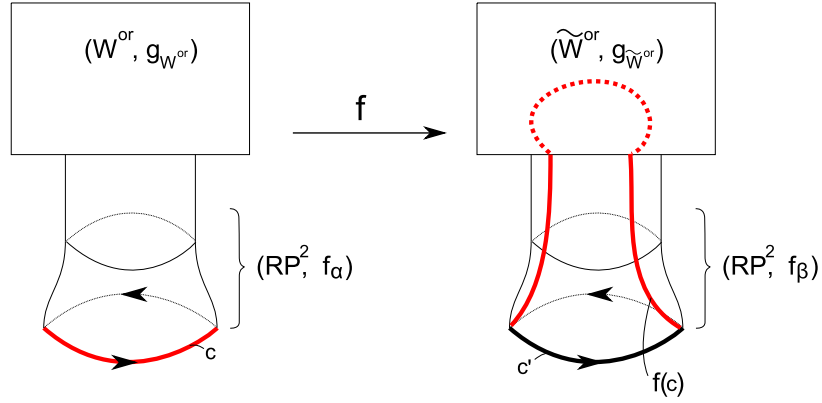

FIGURE 19. Loops $c, f(c)$ and c'.

(II-a): We can show this by using similar arguments in the proof of (I-a).

(II-b): We can show this by using similar arguments in the proof of (I-b). Instead of the mapping class group of $\mathbf{R}P^2$, we use that of the Klein bottle. The mapping class group of a Klein bottle is generated by two elements x and y (see Theorem 4.3 below). We can assume that the cobordism $(\mathbf{R}P^2, f_\alpha) \sharp (\mathbf{R}P^2, f_\beta)$ is given by the right hand side in Figure 21. If f is isotopic to x, we have $\alpha_1 = \beta_2$ and $\alpha_2 = \beta_1$. If f is isotopic to y, we have $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$ (see Figures 22 and 23).

(II-c): We can show this by using similar arguments in the proof of (I-c). $\square$

Theorem 4.3 (Lickorish [4]). *Let K be the Klein bottle. We define a homeomorphism $x \colon K \to K$ as a Dehn twist along the loop c depicted in Figure 24 and a homeomorphism $y \colon K \to K$ as taking the mirror image with respect to the line d depicted in Figure 24. Then the mapping class group of K is generated by the isotopy classes of x and y.*
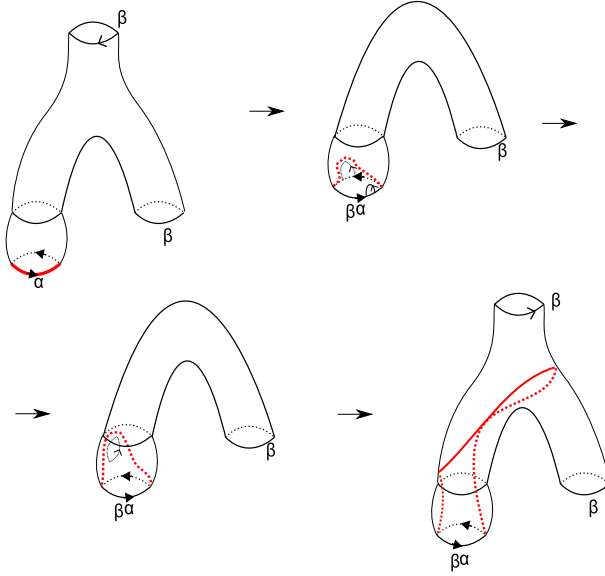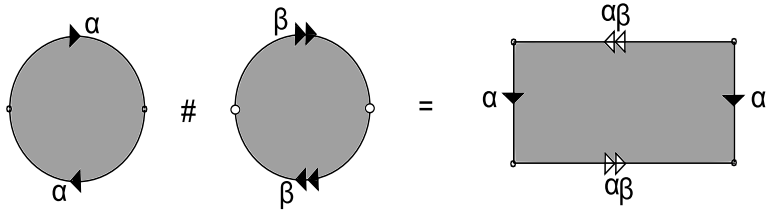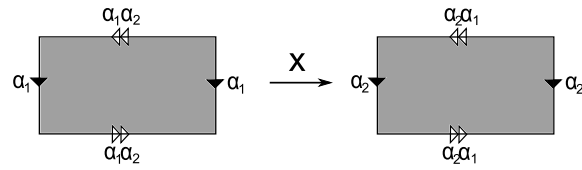
FIGURE 20. A transformation.

FIGURE 21. The cobordism $(\mathbf{R}P^2, f_\alpha)\natural(\mathbf{R}P^2, f_\beta)$.

FIGURE 22. In the case $f = x$.

5. THE AXIOMS OF HQFT

In this section, we check that the pair (A, τ) constructed in Section 4 satisfies the axioms of HQFTs (see Definition 2.5).
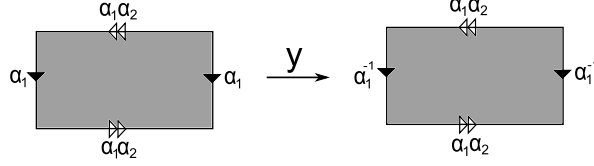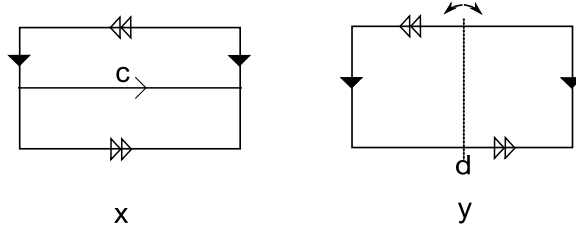
FIGURE 23. In the case $f = y$.

FIGURE 24. Generators of the mapping class group of a Klein bottle.

Proposition 5.1. *The functor τ constructed in Theorem 2.12 from an extended crossed π-algebra $(L, \eta, \varphi, \{\theta_\alpha\}_{\alpha \in \pi}, \Phi)$ is natural with respect to unoriented X-homeomorphisms of unoriented X-manifolds.*

Proof. Let (W_1, M_1, N_1, g_1) and (W_2, M_2, N_2, g_2) be two unoriented X-cobordisms and $f\colon (W_1, M_1, N_1, g_1) \to (W_2, M_2, N_2, g_2)$ be an unoriented X-homeomorphism of unoriented X-cobordisms. If we are given an unoriented X-homeomorphism

$$(W_1, g_1) \cong (W_1^{or}, g_1|_{W_1^{or}}) \sharp (\mathbf{R}P^2, f_{\alpha_1}) \sharp \ldots \sharp (\mathbf{R}P^2, f_{\alpha_n}),$$

then we have

$$(W_2, g_2) \cong (f(W_1^{or}), (g_1|_{W_1^{or}}) \circ f^{-1}) \sharp (\mathbf{R}P^2, f_{\alpha_1} \circ f^{-1}) \sharp \ldots \sharp (\mathbf{R}P^2, f_{\alpha_n} \circ f^{-1}).$$

There is an element $\beta_i \in \pi$ such that f_{β_i} is isotopic to $f_{\alpha_i} \circ f^{-1}$ for all $i = 1, \ldots, n$. It follows from Lemma 4.2 that $\alpha_i = \beta_i$ for all $i = 1, \ldots, n$. From Theorem 2.9, we have $(f|_{N_1})_\sharp \circ \tau(W_1^{or}, g_1|_{W_1^{or}}) = \tau(f(W_1^{or}), (g_1|_{W_1^{or}}) \circ f^{-1}) \circ (f|_{M_1})_\sharp$. This completes the proof. $\square$

Definition 5.2. Let (A, τ) be the pair constructed in Theorem 2.12 from an extended crossed π-algebra $(L, \eta, \varphi, \{\theta_\alpha\}_{\alpha \in \pi}, \Phi)$. An unoriented X-cobordism (W_0, M_0, N) is X-*nice* if for any unoriented X-cobordism (W_1, N', M_1) and unoriented X-homeomorphism $f\colon N \to N'$, we have $\tau(W) = \tau(W_1) \circ f_\sharp \circ \tau(W_0)$, where W is the unoriented X-cobordism obtained from W_1 and W_0 by gluing along f.

In the case where π is trivial, X-niceness is equal to *niceness* introduced in [8]. Then the following lemma is an easy consequence of Theorem 2.9.

Lemma 5.3 ([8])**.** (1) *Let (W_0, M_0, N_0) be an unoriented X-cobordism obtained from two unoriented X-cobordisms (W_0', M_0, N_0') and (W_0'', M_0'', N_0) by gluing along an unoriented X-homeomorphism $g\colon N_0' \to M_0''$. If W_0' and W_0'' are X-nice, so is W_0.*

(2) Let (W_0, M_0, N_0) and (W_1, M_1, N_1) be oriented X-cobordisms and $f \colon N_0 \to M_1$ be an orientation preserving X-homeomorphism. Then we have $\tau(W) = \tau(W_1) \circ f_\sharp \circ \tau(W_0)$, where W is the oriented X-cobordism obtained from W_1 and W_0 by gluing along f.

Proposition 5.4. *The six unoriented X-cobordisms depicted in Figure 25 are X-nice. Hence the pair (A, τ) constructed in Theorem 2.12 from an extended crossed π-algebra $(L, \eta, \varphi, \{\theta_\alpha\}_{\alpha \in \pi}, \Phi)$ satisfies the axioms of Definition 2.5.*
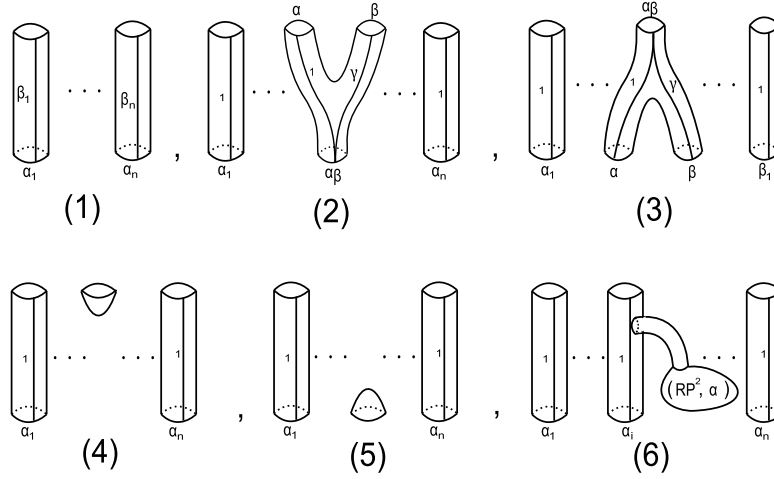

FIGURE 25. Basic X-cobordisms.

Proof. Let (W_0, M_0, N, g) and (W_1, N', M_1, g') be two unoriented X-cobordisms and $f \colon N \to N'$ be an unoriented X-homeomorphism of unoriented X-cobordisms. Let $(W_1 \cup_f W_0, g' \cup_f g)$ be an unoriented cobordism obtained from W_0 and W_1 by gluing along f. Moreover we suppose that we have $(W_1, g') \cong (W_1^{\mathrm{or}}, g'|_{W_1^{\mathrm{or}}}) \sharp (\mathbf{R}P^2, f_{\alpha_1}) \sharp \ldots \sharp (\mathbf{R}P^2, f_{\alpha_n})$.

(I) The case where (W_0, M_0, N, g) is a cobordism depicted in Figure 25 (3).

In this case, we can choose orientations of W_1 and W_0 such that f is an orientation preserving homeomorphism. By Lemma 5.3, we have the following equation:

$$\begin{aligned}
\tau(W_1 \cup_f W_0, g' \cup_f g) &= \psi_{\alpha_1, \cdots, \alpha_n} \circ \tau(W_1^{\mathrm{or}} \cup_f W_0) \\
&= \psi_{\alpha_1, \cdots, \alpha_n} \circ \tau(W_1^{\mathrm{or}}) \circ f_\sharp \circ \tau(W_0) \\
&= \tau(W_1, g') \circ f_\sharp \circ \tau(W_0, g).
\end{aligned}$$

Hence (W_0, M_0, N, g) is X-nice.

(II) The case where (W_0, M_0, N, g) is a cobordism depicted in Figure 25 (1), (4) and (5).

In this case, we can use the same proof as (I).

(III) The case where (W_0, M_0, N, g) is a cobordism depicted in Figure 25 (6).

Suppose that (W_0, M_0, N, g) is given an unoriented X-homeomorphism $(W_0, g) \cong (W_0^{\mathrm{or}}, g|_{W_0^{\mathrm{or}}}) \sharp (\mathbf{R}P^2, f_\alpha)$. Then we have

$$(W_1 \cup_f W_0, g' \cup_f g) \cong$$
$$(W_1^{\mathrm{or}} \cup_f W_0^{\mathrm{or}}, g'|_{W_1^{\mathrm{or}}} \cup_f g|_{W_0^{\mathrm{or}}}) \sharp (\mathbf{R}P^2, f_\alpha) \sharp (\mathbf{R}P^2, f_{\alpha_1}) \sharp \ldots \sharp (\mathbf{R}P^2, f_{\alpha_n}).$$

Furthermore we can give orientations of W_1^{or} and W_0^{or} such that f preserves the orientations. Then we have the following equation:

$$
\begin{aligned}
\tau(W_1 \cup_f W_0, g' \cup_f g) &= \psi_\alpha \circ \psi_{\alpha_1, \cdots \alpha_n} \circ \tau(W_1^{\mathrm{or}} \cup_f W_0^{\mathrm{or}}) \\
&= \psi_\alpha \circ \psi_{\alpha_1, \cdots \alpha_n} \circ \tau(W_1^{\mathrm{or}}) \circ f_\sharp \circ \tau(W_0^{\mathrm{or}}) \\
&= \psi_{\alpha_1, \cdots \alpha_n} \circ \tau(W_1^{\mathrm{or}}) \circ f_\sharp \circ \psi_\alpha \circ \tau(W_0^{\mathrm{or}}) \\
&= \tau(W_1, g') \circ f_\sharp \circ \tau(W_0, g).
\end{aligned}
$$

The third equality follows from Figure 26. The equation in Figure 26 follows from the fact that $f_\sharp =$ id or Φ and that $\Phi(\theta_\alpha) = \theta_\alpha$ for any $\theta_\alpha \in \pi$. Hence (W_0, M_0, N, g) is X-nice.
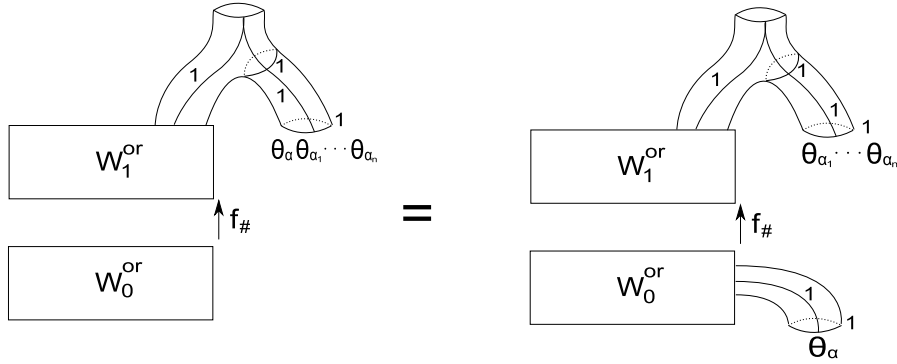

FIGURE 26. $\psi_\alpha \circ \psi_{\alpha_1, \cdots \alpha_n} \circ \tau(W_1^{\mathrm{or}}) \circ f_\sharp \circ \tau(W_0^{\mathrm{or}}) = \psi_{\alpha_1, \cdots \alpha_n} \circ \tau(W_1^{\mathrm{or}}) \circ f_\sharp \circ \psi_\alpha \circ \tau(W_0^{\mathrm{or}})$.

(IV) The case where (W_0, M_0, N, g) is a cobordism depicted in Figure 25 (2).

If we can give orientations of W_1^{or} and W_0 so that f preserves them, we can use the same argument as (I). Suppose that we cannot give such orientations. Then there are an unoriented X-cobordism (W_3, M_3, N_3), an unoriented X-cobordism (W_2, M_2, N_2) and an unoriented X-homeomorphism $f' \colon N_2 \to M_3$ such that $W_1 = W_3 \cup_f W_2$ (see Figure 27), where (W_2, M_2, N_2) is unoriented X-homeomorphic to the unoriented X-cobordism depicted in Figure 25 (3). Let W_4 be an unoriented X-cobordism $W_2 \cup_f W_0$ (see Figure 28). It follows from the proof of Lemma 3.9 that $(W_4, g) \cong (W_4^{\mathrm{or}}, g|_{W_4^{\mathrm{or}}}) \sharp (\mathbf{R}P^2, f_\alpha) \sharp (\mathbf{R}P^2, f_1)$. Moreover it follows from Lemma 3.9 and the definition of extended Frobenius algebra (see Definition 2.10(7)) that

$$
\tau(W_4) = \tau(W_2) \circ f_\sharp \circ \tau(W_0).
$$

Now we have
$(W_3 \cup_{f'} W_4, g' \cup_f g) \cong$
$(W_3^{\mathrm{or}} \cup_{f'} W_4^{\mathrm{or}}, (g' \cup_f g)|_{W_3^{\mathrm{or}} \cup_{f'} W_4^{\mathrm{or}}}) \sharp (\mathbf{R}P^2, f_\alpha) \sharp (\mathbf{R}P^2, f_1) \sharp (\mathbf{R}P^2, f_{\alpha_1}) \sharp \ldots \sharp (\mathbf{R}P^2, f_{\alpha_n})$
and we can give orientations of W_3^{or} and W_4^{or} so that f' preserve them. Hence we
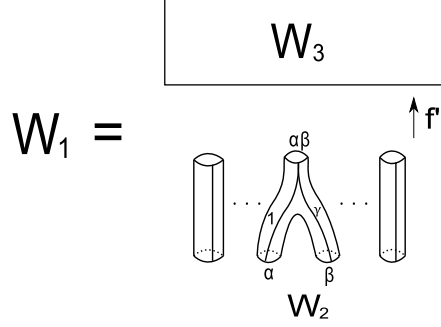
FIGURE 27. $W_1 = W_3 \cup_{f'} W_2$.
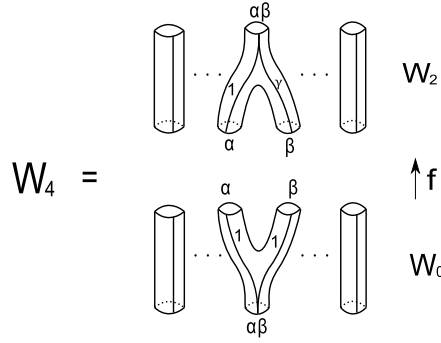
FIGURE 28. $W_4 = W_2 \cup_f W_0$.

have the following equation:

$$\tau(W_1 \cup_f W_4, g' \cup_f g) = (W_3 \cup_{f'} W_4, g' \cup_f g)$$
$$= \psi_{\alpha_1, \cdots \alpha_n} \circ \psi_\alpha \circ \circ \psi_1 \circ \tau(W_3^{\mathrm{or}} \cup_{f'} W_4^{\mathrm{or}})$$
$$= \psi_{\alpha_1, \cdots \alpha_n} \circ \psi_\alpha \circ \circ \psi_1 \circ \tau(W_3^{\mathrm{or}}) \circ f'_\sharp \circ \tau(W_4^{\mathrm{or}})$$
$$= \psi_{\alpha_1, \cdots \alpha_n} \circ \tau(W_3^{\mathrm{or}}) \circ f'_\sharp \circ \psi_\alpha \circ \psi_1 \circ \tau(W_4^{\mathrm{or}})$$
$$= \tau(W_3) \circ f'_\sharp \circ \tau(W_4)$$
$$= \tau(W_3) \circ f'_\sharp \circ \tau(W_2) \circ f_\sharp \circ \tau(W_0)$$
$$= \tau(W_1) \circ f_\sharp \circ \tau(W_0).$$

Hence W_0 is X-nice.   □

6. EXAMPLES

In this section, we construct examples of HQFTs and extended crossed group algebras.

Firstly we will construct an example of unoriented HQFTs.

Example 6.1. This construction is similar to "primitive cohomological HQFT" constructed by Turaev [6] and his construction is inspired by the work of Freed

and Quinn [3]. Let π be $\mathbf{Z}/2\mathbf{Z}$ and X be a $K(\pi, 1)$ space (in particular X is homotopy equivalent to $\mathbf{R}P^\infty$). Given $d \geq 0$ we take a $(d+1)$-dimensional cocycle $\theta \in C^{d+1}(X; R^\times)$, where $R^\times$ is the unit group of R. For any unoriented d-dimensional X-manifold (M, g), we define an R-module $A(M, g)$ by Rv_a, where $a \in C_d(M; \mathbf{Z}/2\mathbf{Z})$ is a fundamental cycle and Rv_a is the free R-module of rank 1 generated by v_a. If $a, b \in C_d(M; \mathbf{Z}/2\mathbf{Z})$ are two fundamental cycles, then we give the relation $v_a = g^*(\theta)(c)v_b$, where c is a $(d+1)$-dimensional singular chain in M such that $\partial c = a + b$. The element $g^*(\theta)(c) \in R^\times$ does not depend on the choice of c. For any unoriented X-homeomorphism $f\colon (M, g) \to (M', g')$, we define an R-homomorphism $f_\sharp\colon A(M, g) \to A(M', g')$ by $f_\sharp(v_a) = v_{f_*(a)}$.

Let (W, M_0, M_1, g) be an unoriented $(d+1)$-dimensional X-cobordism. Take a cycle $B \in C_{d+1}(W, \partial W; \mathbf{Z}/2\mathbf{Z})$ such that $[B] \in H_{d+1}(W, \partial W; \mathbf{Z}/2\mathbf{Z})$ is the fundamental class. Then we have $\partial B = a_0 + a_1$, where $\partial\colon C_{d+1}(W, \partial W; \mathbf{Z}/2\mathbf{Z}) \to C_d(M_0; \mathbf{Z}/2\mathbf{Z}) \oplus C_d(M_1; \mathbf{Z}/2\mathbf{Z})$ is the connected homomorphism and $a_0 \in C_d(M_0; \mathbf{Z}/2\mathbf{Z})$, $a_1 \in C_d(M_1; \mathbf{Z}/2\mathbf{Z})$ are fundamental cycles. Then we define an R-homomorphism $\tau(W, g)\colon A(M_0, g_{M_0}) \to A(M_1, g_{M_1})$ by $\tau(W, g)(v_{a_0}) = (g^*(\theta)(B))^{-1}v_{a_1}$. The map $\tau(W, g)$ does not depend on the choice of B.

The pair (A, τ) is an unoriented $(d+1)$-dimensional HQFT with target X. Moreover the isomorphism class (A, τ) does not depend on the choice of a singular cocycle representation θ of the homology class $[\theta] \in H^{d+1}(X, ; R^\times)$. For any closed unoriented d-dimensional X-cobordism (W, g), the map $\tau(W, g)$ is an involution.

Secondly we make an example of extended crossed group algebras below.

Example 6.2. Let π be the group $\mathbf{Z}/2\mathbf{Z} = \{1, -1\}$ and $\{l_\alpha\}_{\alpha \in \pi}$ be a set whose index set is π. Let $\{\kappa_{\alpha,\beta} \in R^\times\}_{\alpha,\beta \in \pi}$ be a normalized 2-cocycle, that is, $\kappa_{1,1} = 1$ and $\kappa_{\alpha,\beta}\kappa_{\alpha\beta,\gamma} = \kappa_{\alpha,\beta\gamma}\kappa_{\beta,\gamma}$, where $R^\times$ is the group of units of R. Note that for any $\alpha \in \pi$, we have $\kappa_{1,\alpha} = \kappa_{\alpha,1} = 1$.

For any $\alpha \in \pi$, let L_α be the free R-module of rank 1 generated by l_α, that is, $L_\alpha = Rl_\alpha$. Put $L = L_1 \oplus L_{-1}$. Multiplication of L is defined by $l_\alpha l_\beta = \kappa_{\alpha,\beta}l_{\alpha\beta}$. A bilinear form $\eta\colon L \otimes L \to R$ is defined by $\eta(l_\alpha \otimes l_\alpha) = \kappa_{\alpha,\alpha}$ for any $\alpha \in \pi$ and $\eta(l_\alpha \otimes l_\beta) = 0$ for $\beta \neq \alpha$. For any $\beta \in \pi$, put $\varphi_\beta = \mathrm{id}$. Take an element $a \in R$ which satisfies $a^2 = 1$ and put $\theta_\alpha = al_1$ for any $\alpha \in \pi$. Then $(L = \bigoplus L_\alpha, \eta, \varphi, \{\theta_\alpha\}_{\alpha \in \pi}, \Phi)$ is an extended crossed π-algebra.

We can easily prove that the algebra $(L = \bigoplus L_\alpha, \eta, \varphi, \{\theta_\alpha\}_{\alpha \in \pi}, \Phi)$ satisfies the axioms in Definition 2.10 except $(3), (7)$ and (11). Since we have $l_\alpha l_\beta = \kappa_{\alpha,\beta}l_{\alpha\beta} = \kappa_{\alpha,\beta}l_{\beta\alpha} = \kappa_{\alpha,\beta}\kappa_{\beta,\alpha}^{-1}l_\beta l_\alpha = l_\beta l_\alpha$, L satisfies the axiom (3).

To check the axiom (7), we need to compute $\Delta_{\alpha,\beta}(l_{\alpha\beta})$ for any $\alpha, \beta \in \pi$. Put $\Delta_{\alpha,\beta}(l_{\alpha\beta}) = kl_\alpha \otimes l_\beta$. Then we have

$$(6.1) \qquad\qquad (\mathrm{id} \otimes \eta) \circ (\Delta_{\alpha,\beta} \otimes \mathrm{id})(l_{\alpha\beta} \otimes l_\beta) = l_{\alpha\beta}l_\beta$$

(see Figure 4). The left hand side of (6.1) is equal to $k\kappa_{\beta,\beta}l_\alpha$ and the right hand side is equal to $\kappa_{\alpha\beta,\beta}l_{\alpha\beta\beta} = \kappa_{\alpha\beta,\beta}l_\alpha$. Hence $k = \kappa_{\beta,\beta}^{-1}\kappa_{\alpha\beta,\beta}$ and we have

$$m \circ (\Phi \otimes \varphi_\gamma) \circ \Delta_{\alpha,\beta}(l_{\alpha\beta}) = k\kappa_{\alpha,\beta}l_{\alpha\beta}$$
$$= l_{\alpha\beta}.$$

To check the axiom (11), we need to compute $q(1) \in L_1$. We consider $\tau^L(Q')$, where the cobordism Q' is depicted in Figure 29 whose bottom base is empty and

whose top base is $(\mathbf{S}^1, \alpha\beta) \sqcup (\mathbf{S}^1, \alpha\beta)$. Put $\tau^L(Q')(1) = k'l_{\alpha\beta} \otimes l_{\alpha\beta}$. Now we have

$$(6.2) \qquad (\mathrm{id} \otimes \eta) \circ (\tau(Q') \otimes \mathrm{id})(l_{\beta\alpha}) = \varphi_{\beta\gamma}(l_{\beta\alpha})$$

(see Figure 5). The left hand side of (6.2) is equal to $k'\kappa_{\alpha\beta,\alpha\beta}l_{\alpha\beta}$ and the right hand side is equal to $l_{\alpha\beta}$. Hence $k' = \kappa_{\alpha\beta,\alpha\beta}^{-1}$ and we have $q(1) = m \circ \tau^L(Q')(1) = k'l_{\alpha\beta}l_{\alpha\beta} = l_1$. This shows that L satisfies the axiom (11).

Remark 6.3. Note that Turaev [6] shows that the algebra $(L = \bigoplus L_\alpha, \eta, \varphi)$ is a crossed π-algebra.
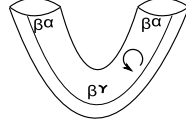

FIGURE 29. The cobordism $(Q', \emptyset, (\mathbf{S}^1, \alpha\beta) \sqcup (\mathbf{S}^1, \alpha\beta))$.

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